Engineering Notes

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Use of a Variational Method for Solving Viscoelastic Stability Problems

DEAN T. MOOK*
Virginia Polytechnic Institute and State University, Blacksburg, Va.

AND

RICHARD P. BENEDICT†
Naval Ordnance Laboratory, White Oak, Md.

THE stability of parallel, or nearly parallel, flows has been and will be the subject of many investigations. To date, the neutral stability curves for these flows have been determined by either a finite-difference or an asymptotic procedure. For the finite-difference procedures, a relatively large amount of computing time and an accurate initial guess for the solution are required. For the asymptotic procedures, the results lack accuracy, and the method has a limited range of application and generally requires considerable analysis.

Lee and Reynolds[1] have used a variational method to obtain some solutions to the Orr-Sommerfeld problem for plane Poiseuille flow. These solutions, which they compared with those of Thomas,[2] do not correspond to the neutrally stable case. However, the method can be used to obtain neutral stability curves with considerable accuracy and with considerably less computing time than required by the finite-difference methods. Moreover, this variational method requires little analysis and is relatively simple to program.

In the present Note, this variational method is modified slightly so that points on the neutral stability curve are found directly, and viscoelastic terms are included in the development of the characteristic determinant. The results for plane Poiseuille flow of viscoelastic liquids are compared with those obtained by the other two methods.

An outline of the variational method follows. For convenience, the stability equation is denoted by

$$L\phi = 0 \tag{1a}$$

and the set of boundary conditions by

$$B\phi = 0 \tag{1b}$$

One can obtain an adjoint operator L^* and a set of adjoint boundary conditions B^* by integrating the following equation by parts:

$$\int_{y_1}^{y_2} uLvdy = \int_{y_1}^{y_2} vL^*udy \tag{2}$$

In Eq. (2) u is any function satisfying $Bu = 0$, and v is any function satisfying $B^*v = 0$. Then it can be shown that the integral I, defined by

$$I = \int_{y_1}^{y_2} \phi^* L\phi dy \tag{3}$$

is zero and stationary with respect to variations in ϕ^* and ϕ, when ϕ^* and ϕ are solutions of

$$L^*\phi^* = 0 \tag{4a}$$

$$B^*\phi^* = 0 \tag{4b}$$

(the adjoint problem) and Eq. (1), respectively, and when the variations satisfy the respective boundary conditions. For the details, the reader is referred to the paper by Lee and Reynolds.

Chun and Schwarz[3] as well as Walters,[4] using different constitutive equations, have found the stability equation for parallel flows of viscoelastic liquids to be

$$(u - c)(D^2 - \alpha^2)\phi - (D^2u)\phi = 1/i\alpha Re\ [1 - i\alpha\beta(u - c)](D^2 - \alpha^2)^2\phi + (\beta/Re)(D^4u)\phi \tag{5a}$$

and, when the flow is between plates located at $y = \pm 1$, the boundary conditions are

$$\phi(\pm 1) = 0 \text{ and } D\phi(\pm 1) = 0 \tag{5b}$$

For an explanation of the symbols, the reader is referred to Chun and Schwarz, or to Walters.

The corresponding adjoint problem is

$$L^*\phi^* = [1 - i\alpha\beta(u - c)](D^2 - \alpha^2)\phi^* - i\alpha\beta[4D(D^2uD\phi^*) + 2D^2u(D^2 - \alpha^2)\phi^* + 4D(D^2 - \alpha^2)D\phi^*] - i\alpha RE[(u - c)\ (D^2 - \alpha^2)\phi^* + 2DuD\phi^*] = 0 \tag{6a}$$

$$\phi^*(\pm 1) = 0, \text{ and } D\phi^*(\pm 1) = 0 \tag{6b}$$

The familiar Rayleigh-Ritz scheme is to assume solutions ϕ and ϕ^* in the form of a series of functions; i.e., assume

$$\phi = \sum_{n=1}^{N} A_n f_n(y) \tag{7a}$$

and

$$\phi^* = \sum_{n=1}^{N} A_n^* f_n^*(y) \tag{7b}$$

Satisfaction of the boundary conditions can be assured by making each term in the series satisfy them. This is usually done (but it is not necessary). Since the boundary conditions on ϕ and ϕ^* are the same, one can choose $f_n = f_n^*$. When this is done and Eq. (7) substituted into Eq. (3), one finds that I will have a stationary value if

$$\det\{I_{mn}^{(2)} + 2\alpha^2 I_{mn}^{(1)} + 2^4 I_{mn}^{(0)} + i\alpha Re J_{mn}^{(2)} + J_{mn}^{(1)} + \alpha^2 J_{mn}^{(0)} - c[I_{mn}^{(1)} + \alpha^2 I_{mn}^{(0)}] - i\alpha\beta J_{mn}^{(3)} + 2\alpha^2 J_{mn}^{(2)} + \alpha^4 J_{mn}^{(0)} - c[I_{mn}^{(2)} + 2\alpha^2 I_{mn}^{(1)} + \alpha^4 I_{mn}^{(0)}]\} = 0 \tag{8a}$$

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* Assistant Professor, Department of Engineering Mechanics. Member AIAA.

† Mechanical Engineer, Underwater Mechanical Engineering Department.

Table 1 Variation in α and Reynolds number with the number of terms in Eq. (7)

$c = 0.20 \quad \beta = 0$		
No. of terms	α	Reynolds no.
9	0.74723	11368
13	0.75483	12059
17	0.75478	12117
19	0.75555	12124
21	0.75555	12124

where

$$J_{mn}^{(3)} = \int_{-1}^{+1} uf_m D^4 f_n dy \tag{8b}$$

and all other Imn and Jmn are as defined by Lee and Reynolds. For a given choice of functions in Eq. (7), it is assumed that the combination that produces a stationary value for I is the best combination attainable with those functions.

One may assign values to α, Re, and β, considering c to be the (complex) eigenvalue of the determinant. There are several IBM subroutines available that can be used to find c. Or one may assign values to β and to the real and imaginary parts of c and then solve for the corresponding α and Re. The former was done by Lee and Reynolds and the latter was done here. For neutral stability $c_i = 0$; thus real values were assigned to both c and β. Initial guesses were made for α and Re, and then increments in these values were obtained by solving

$$\frac{\partial R(\text{det})}{\partial \alpha} \Delta\alpha + \frac{\partial R(\text{det})}{\partial Re} \Delta Re + R(\text{det}) = 0 \tag{9a}$$

and

$$\frac{\partial I(\text{det})}{\partial \alpha} \Delta\alpha + \frac{\partial I(\text{det})}{\partial Re} \Delta Re + I(\text{det}) = 0 \tag{9b}$$

In Eq. (9) R(det) and I(det) are the real and imaginary parts, respectively, of the determinant given in Eq. (8).

Table 2 Variations in α and Reynolds number with the number of iterations

$c = 0.21 \quad \beta = 0$		
No. of iterations	α	Reynolds no.
0	0.75555	12141
1	0.75977	11543
2	0.76873	11003
3	0.78013	10607
4	0.78701	10441
5	0.78803	10419
6	0.78804	10418

Table 3 Comparison of critical Reynolds number, α, and c for various β

β	C & S	M & G	Present
		Re	
0	5775	5399	5770
0.1	5537	5170	5542
0.5	4620	4186	4707
1.0	3630	2662	3855
		α	
0	1.026	1.022	1.021
0.1	1.026	1.033	1.030
0.5	1.08	1.088	1.071
1.0	1.16	1.226	1.125
		c	
0	0.2646	0.2672	0.2640
0.1	0.2668	0.2708	0.2672
0.5	0.2840	0.2886	0.2808
1.0	0.3065	0.3296	0.2990

Lee and Reynolds found that $f_n = f_n^* = (1 - y^2)^2 y^{2(n-1)}$ worked well in the sense that only a few terms in Eq. (7) are needed. These functions were also used here.

Table 1 shows the convergence of α and Re as the number of terms in Eq. (7) is increased, and Table 2 shows the convergence of α and Re with the number of iterations for 21 terms. These are typical of the cases investigated. Finally, in Table 3, the critical Reynolds number and the corresponding c and α for various values of β are compared with the results obtained by Mook and Graebel[5] using an asymptotic approach, and with the results obtained by Chun and Schwarz using a finite-different procedure. There is close agreement between the results of Chun and Schwarz and the present results. Yet the computing time for the present results is one-fifth to one-fourth of the computing time for a finite-difference method used by the authors. The amazing ease with which the present results were found and their accuracy suggest trying this variational method on other problems of the Orr-Sommerfeld type.

References

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